INVERSE INDEFINITE SPECTRAL PROBLEM FOR SECOND ORDER DIFFERENTIAL OPERATOR WITH COMPLEX PERIODOC COEFFICIENTS

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ABSTRACT.

The inverse problem for the Sturm- Liouville operator with complex periodic potential and discontinuous coefficients on the axis is studied. Main characteristics of the fundamental solutions are investigated, the spectrum of the operator is studied. We give formulation of the inverse problem, prove a uniqueness theorem and provide a constructive procedure for the solution of the inverse problem.

Key words:

Discontinuous equations; Tuning points; Spectral singularities; Inverse spectral problems; Continuous spectrum;

MSC:

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INTRODUCTION.

We consider the differential equation

$$-y''(x) + q(x)\,y(x) = \lambda^2 \rho(x)\,y(x) \tag{1}$$

in the space $L_2(-\infty, +\infty)$ where the prime denotes the derivative with respect to the space coordinate and assume that the potential $q(x)$ is of the form

$$q(x) = \sum_{n=1}^{\infty} q_n e^{inx}, \tag{2}$$

the condition$\sum_{n=1}^{\infty} |q_n|^2 = q < \infty$ is satisfies, λ is a complex number, and

$$\rho(x) = \begin{cases} 1 & for \quad x \geq 0, \\ -\beta^2 & for \quad x < 0. \end{cases} \tag{3}$$

The function $\rho(x)$ is called density function, and the function $q(x)$ is called the potential function of equation (1). If$\rho(x)=1$, then equation (1) is called the potential equation. The potential equation often is met in physical, technical and astronomical problems. Generalized Legendre equation, degenerate hypergeometrical equation, Bessel's equation and also Mathieu equation after suitable substitution coincide with potential equation (1)[1,p.374]-[2].
As a rule, such problems are connected with discontinuous properties of materials. Inverse problems of spectral analysis consist of recovering operators from their spectral characteristics.
In this paper we will study the spectrum and also solve the inverse problem for singular non-self-adjoint operator. As the coefficient allows bounded analytic continuation to the upper half-plane of the complex plane$z=x+it$, we can conduct detailed analysis of problem (1)-(3).
Our investigation was stimulated by M.G.Casymov's paper [3] where he first considered this potential, and his co-workers [4]. Especially, we would like to indicate the paper V. Guillemin, A. Uribe [5] where the potential $q(x)=\sum_{n=1}^{\infty} q_n e^{inx}$ plays a vital part for solving the KdV equation. Later in 1990 the results obtained in [3] were extended by Pastur L.A., Tkachenko V.A [6].
The inverse problems connected with potentials of the form $q(x)=\sum_{n=1}^{\infty} q_n e^{inx}$ where$\sum_{n=1}^{\infty}|q_n|=q<\infty$, were considered in [7-9].
The paper consists of three parts.
In part 1 we study the properties of fundamental system of solutions of equation (1). The spectrum of problem (1)-(3) is investigated in part 2. In part 3 we give a formulation of the inverse problem, prove a uniqueness theorem and provide a constructive procedure for the solution of the inverse problem.

1. REPRESENTATION OF FUNDAMENTAL SOLUTIONS.

Here we study the solutions of the main equation

$$-y''(x)+q(x)\,y(x)=\lambda^2\rho(x)\,y(x)$$

that will be convenient in future.
We first consider the solutions $f_1^+(x,\lambda)$ and $f_2^+(x,\lambda)$, determined by the conditions at infinity

$$\lim_{Imx\to\infty} f_1^+(x,\lambda)\,e^{-i\lambda x}=1,$$

$$\lim_{Imx\to\infty} f_2^+(x,\lambda)\,e^{-\beta\lambda x}=1.$$

We can prove the existence of these solutions if the condition $\sum_{n=1}^{\infty}|q_n|^2=q<\infty$ is fulfilled for the potential. This will be unique restriction on the potential and later on we'll consider it to be fulfilled.
Theorem 1. Let $q(x)$ be of the form (2) and $\rho(x)$ satisfy condition (3). Then equation (1) has special solutions of the form

$$f_1^{+}(x,\lambda)=e^{i\lambda x}\left(1+\sum_{n=1}^{\infty}\frac{1}{n+2\lambda}\sum_{\alpha=n}^{\infty}V_{n\alpha}e^{i\alpha x}\right),\quad for\ x\geq 0, \tag{4}$$

$$f_2^{+}(x,\lambda)=e^{\lambda\beta x}\left(1+\sum_{n=1}^{\infty}\frac{1}{n-2i\lambda\beta}\sum_{\alpha=n}^{\infty}V_{n\alpha}e^{i\alpha x}\right),\quad for\ \ x<0. \tag{5}$$

where the numbers $V_{n\alpha}$ are determined from the following recurrent relations

$$\alpha(\alpha-n)V_{n\alpha}+\sum_{s=n}^{\alpha-1}q_{\alpha-s}V_{ns}=0,\qquad 1\leq n<\alpha, \tag{6}$$

$$\alpha\sum_{n=1}^{\alpha}V_{n\alpha}+q_{\alpha}=0, \tag{7}$$

and the series

$$\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\alpha=n}^{\infty}\alpha\left|V_{n\alpha}\right| \tag{8}$$

converges.
The proof of the theorem is similar to the proof of [6] and therefore we don't cite it here.
Remark1: If $\lambda\neq-\frac{n}{2}$ and$Im\lambda\geq 0$, then$f_1^{+}\left(x,\lambda\right)\in L_2\left(0,+\infty\right)$.
Remark2: If $\lambda\neq-\frac{in}{2\beta}$ and$Re\lambda\geq 0$, then$f_2^{+}\left(x,\lambda\right)\in L_2\left(-\infty,0\right)$.
Extending $f_1^{+}\left(x,\lambda\right)$and $f_2^{+}\left(x,\lambda\right)$ as solutions of equation (1) on$x<0$ and $x\geq 0$ respectively and using the conjunction conditions

$$\begin{array}{l} y\left(0+\right)=y(0-),\\ y'\left(0+\right)=y'(0-), \end{array} \tag{9}$$

we can prove the following lemma.

Lemma 1: $f_1^{+}\left(x,\lambda\right)$ and $f_2^{+}\left(x,\lambda\right)$ may be extended as solutions of equation (1) on $x<0$ and$x\geq 0$, respectively. Then we get

$$f_2^{+}\left(x,\lambda\right)=C_{11}\left(\lambda\right)f_1^{+}\left(x,\lambda\right)+C_{12}\left(\lambda\right)f_1^{-}\left(x,\lambda\right)\ \ for\ \ x\geq 0,$$

$$f_1^{+}\left(x,\lambda\right)=C_{22}\left(\lambda\right)f_2^{+}\left(x,\lambda\right)+C_{21}\left(\lambda\right)f_2^{-}\left(x,\lambda\right),\quad for\ \ x<0\ ,$$

where

$$f_{1,2}^{-}\left(x,\lambda\right)=f_{1,2}^{+}\left(x,-\lambda\right),$$

$$C_{11}\left(\lambda\right)=\frac{W[f_2^{+}\left(0,\lambda\right),f_1^{-}\left(0,\lambda\right)]}{2i\lambda}, \tag{10}$$

$$C_{12}(\lambda) = \frac{W[f_1^+(0,\lambda), f_2^+(0,\lambda)]}{2i\lambda},$$

$$C_{22}(\lambda) = \frac{i}{\beta}C_{11}(-\lambda), C_{21}(\lambda) = -\frac{i}{\beta}C_{12}(\lambda). \tag{11}$$

Proof: It is easy to see that equation (1) has fundamental solutions $f_1^+(x,\lambda)$,$f_1^-(x,\lambda)$ ($f_2^+(x,\lambda)$,$f_2^-(x,\lambda)$) on the $|Im\lambda| < \frac{\varepsilon}{2}$ ($|Re\lambda| < \frac{\varepsilon}{2}$), for which

$$W\left[f_1^+(x,\lambda), f_1^-(x,\lambda)\right] = 2i\lambda,$$

$$W\left[f_2^+(x,\lambda), f_2^-(x,\lambda)\right] = 2\lambda\beta,$$

is satisfied

Really, since $W[f_1^+(x,\lambda), f_1^-(x,\lambda)]$ and $W[f_2^+(x,\lambda), f_2^-(x,\lambda)]$ are independent of x and the functions $f_1^+(x,\lambda)$,$f_1^-(x,\lambda)$ and $f_2^+(x,\lambda)$,$f_2^-(x,\lambda)$ allow holomorphic continuation on x to upper and lower half-planes, respectively, the Wronskian coincides as $Imx \to \infty$. We can show that

$$\lim_{Imx\to\infty} f_1^{\pm(j)}(x,\lambda)\, e^{\mp i\lambda x} = (\pm i\lambda)^j \qquad j = 0, 1, \tag{12}$$

$$\lim_{Imx\to\infty} f_2^{\pm(j)}(x,\lambda)\, e^{\mp \lambda x} = (\pm \lambda\beta)^j \qquad j = 0, 1. \tag{13}$$

So that

$$W\left[f_1^+(x,\lambda), f_1^-(x,\lambda)\right] = 2i\lambda,$$

$$W\left[f_2^+(x,\lambda), f_2^-(x,\lambda)\right] = 2\lambda\beta.$$

Then each solution of equation (1) may be represented as linear combinations of these solutions.

$$f_2^+(x,\lambda) = C_{11}(\lambda) f_1^+(x,\lambda) + C_{12}(\lambda) f_1^-(x,\lambda) \quad for \;\; x \geq 0.$$

$$f_1^+(x,\lambda) = C_{22}(\lambda) f_2^+(x,\lambda) + C_{21}(\lambda) f_2^-(x,\lambda)\,, \qquad for \;\; x < 0\,,$$

Using the conjunction conditions (9) it is easy to obtain the relation (10-11).
Let

$$f_n^{\pm}(x) = \lim_{\lambda\to\mp\frac{n}{2}} (n \pm 2\lambda) f_1^{\pm}(x,\lambda) = \sum_{\alpha=n}^{\infty} V_{n\alpha} e^{i\alpha x} e^{-i\frac{n}{2}x}, \tag{14}$$

It follows from relation (6) that if $V_{nn} \neq 0$, then $V_{n\alpha} \neq 0$ for all $\alpha > n$ and therefore $f_n^{\pm}(x) \neq 0$. Consequently, the points $\pm\frac{n}{2}$, $n \in N$ are not singular points for $f_1^{\pm}(x,\lambda)$.

Then $W[f_n^{\pm}(x), f_1^{\mp}\left(x, \mp\frac{n}{2}\right)] = 0$ and consequently the functions $f_n^{\pm}(x), f_1^{\mp}\left(x, \mp\frac{n}{2}\right)$, that are solutions of equation (1) for $\lambda = \pm\frac{n}{2}$, are linear dependent.
Therefore

$$f_n^{\pm}(x) = V_{nn} f_1^{\mp}\left(x, \mp\frac{n}{2}\right), \tag{15}$$

2.1. SPECTRUM OF OPERATOR L.

Let L be an operator generated by the operation $\frac{1}{\rho(x)}\left\{-\frac{d^2}{dx^2} + q(x)\right\}$ in the space $L_2(-\infty, +\infty, \rho(x))$.
To study the spectrums of the operator L at first we calculate the kernel of the resolvent of the operator $\left(L - \lambda^2 I\right)$ by means of general methods.
To construct the kernel of the resolvent of operator L, we consider the equation

$$-y''(x) + q(x)\, y(x) = \lambda^2 \rho(x)\, y(x) + f(x).$$

Here, $f(x)$ is an arbitrary function belonging to $L_2(-\infty, +\infty)$. Divide the plane λ into sectors

$$S_k = \{{}^{k\pi}\!/_2 < \arg\lambda < {}^{(k+1)\pi}\!/_2\}, k = \overline{0,3}.$$

When $\lambda \in S_0$, we note that every solution of equation (1) can be written in the form

$$y(x,\lambda) = C_1(x,\lambda)\, f_1^+(x,\lambda) + C_2(x,\lambda)\, f_2^+(x,\lambda) \quad . \tag{16}$$

Using the method of variation of constant, we obtain that

$$C_1^{/}(x,\lambda) = -\frac{1}{W[f_1^+, f_2^+]}\rho(x)\, f_2^+(x,\lambda)\, f(x)$$

$$C_2^{/}(x,\lambda) = \frac{1}{W[f_1^+, f_2^+]}\rho(x)\, f_1^+(x,\lambda)\, f(x)$$

By virtue of the condition $y(x,\lambda) \in L_2(-\infty, +\infty)$, we find that

$$C_2(\infty,\lambda) = C_1(-\infty,\lambda) = 0.$$

Consequently, we have

$$C_1(x,\lambda) = \int_{-\infty}^{x} \frac{1}{W[f_1^+, f_2^+]}\rho(t)\, f_2^+(t,\lambda)\, f(t)\, dt$$

$$C_2(x,\lambda) = -\int_{x}^{\infty} \frac{1}{W[f_1^+, f_2^+]}\rho(t)\, f_1^+(t,\lambda)\, f(t)\, dt.$$

Substitute them in (16) we get

$$y(x,\lambda) = \int_{-\infty}^{\infty} R_{11}(x,t,\lambda)\, \rho(t)\, f(t)\, dt$$

where

$$R_{11}(x,t,\lambda) = \frac{1}{W[f_1^+, f_2^+]} \begin{cases} f_1^+(x,\lambda)\, f_2^+(t,\lambda) & for\ t < x \\ f_1^+(t,\lambda)\, f_2^+(x,\lambda) & for\ t > x \end{cases} \quad \lambda \in S_0. \qquad (17)$$

Calculating analogously we can construct the kernel of the resolvent on the sectorsS_k, $\ k = \overline{1,3}$, namely

$$R_{12}(x,t,\lambda) = \frac{1}{W[f_1^+, f_2^-]} \begin{cases} f_1^+(x,\lambda)\, f_2^-(t,\lambda) & for\ t < x \\ f_1^+(t,\lambda)\, f_2^-(x,\lambda) & for\ t > x \end{cases} \quad \lambda \in S_1. \qquad (18)$$

$$R_{21}(x,t,\lambda) = \frac{1}{W[f_1^-, f_2^-]} \begin{cases} f_1^-(x,\lambda)\, f_2^-(t,\lambda) & for\ t < x \\ f_1^-(t,\lambda)\, f_2^-(x,\lambda) & for\ t > x \end{cases} \quad \lambda \in S_2. \qquad (19)$$

$$R_{22}(x,t,\lambda) = \frac{1}{W[f_1^-, f_2^+]} \begin{cases} f_1^-(x,\lambda)\, f_2^+(t,\lambda) & for\, t < x \\ f_1^-(t,\lambda)\, f_2^+(x,\lambda) & for\, t > x \end{cases} \quad \lambda \in S_3. \qquad (20)$$

<u>Lemma 2.</u> L has no eigenvalues for real and pure imaginary λ. It's continuous spectra consist of axes $Re\lambda = 0$ and $Im\lambda = 0$ on which there may exist spectral singularities coinciding with the numbers $\frac{in}{2\beta}$, $\ \frac{n}{2}$, $\ n = \pm 1, \pm 2, \pm 3, ...$

Proof: We recall that equation (1) has fundamental solutions$f_1^+(x,\lambda)$,$f_1^-(x,\lambda)$ ($f_2^+(x,\lambda)$,$f_2^-(x,\lambda)$) on$|Im\lambda| < \frac{\varepsilon}{2}$ $\ (|Re\lambda| < \frac{\varepsilon}{2})$, for which

$$W\left[f_1^+(x,\lambda), f_1^-(x,\lambda)\right] = 2i\lambda,$$

$$W\left[f_2^+(x,\lambda), f_2^-(x,\lambda)\right] = 2\lambda\beta,$$

is satisfied

Then for $Im\lambda = 0$ solution of equation (1) can be written in the form

$$y(x,\lambda) = C_1 f_1^+(x,\lambda) + C_2 f_1^-(x,\lambda)$$

In case $Im\lambda = 0$ the solution $f_1^{\pm}(x,\lambda)$ has the form

$$f_1^{\pm}(x,\lambda) = e^{\pm iRe\lambda x}\left(1 + \sum_{n=1}^{\infty} \frac{1}{n \pm 2\lambda} \sum_{\alpha=n}^{\infty} V_{n\alpha} e^{i\alpha x}\right)$$

then $y(x,\lambda) \in L_2(-\infty, +\infty)$ except when $C_1 = C_2 = 0$.

Analogously we can prove case$Re\lambda = 0$. Since in $|Re\lambda| < \frac{\varepsilon}{2}$ the functions$f_2^+(x,\lambda)$, $f_2^-(x,\lambda)$ form fundamentals solutions, then

$$y(x,\lambda) = C_3 f_2^+(x,\lambda) + C_4 f_2^-(x,\lambda) \quad .$$

If $Re\lambda = 0$ then the solution $f_2^{\pm}(x,\lambda)$ has the form

$$f_2^{\pm}(x,\lambda)=e^{\pm iIm\lambda\beta x}\left(1+\sum_{n=1}^{\infty}\frac{1}{n\pm 2Im\lambda\beta}\sum_{\alpha=n}^{\infty}V_{n\alpha}e^{i\alpha x}\right)$$

then $y\left(x,\lambda\right)\in L_2\left(-\infty,+\infty\right)$ except when $C_3=C_4=0$.
In order all numbers from the axes $\{\lambda : Re\lambda=0\}$ and $\{\lambda : Im\lambda=0\}$ belong to the continuous spectra it suffices to show that domain of value of the operator $\left(L-\lambda^2 I\right)$ is dense in $L_2\left(-\infty,+\infty\right)$, so that the orthogonal complement of the set $R\left(x,t,\lambda\right)$ consists of only zero element .
Let $\psi\left(x\right)\in L_2\left(-\infty,+\infty\right)$, $\psi\left(x\right)\neq 0$ and

$$\int_{-\infty}^{+\infty}\left(Lf-\lambda^2 f\right)\overline{\psi\left(x\right)}dx=0 \tag{21}$$

be satisfied for any $f\left(x\right)\in D\left(L\right)$.
From (21) it follows that $\psi\left(x\right)\in D\left(L^*\right)$ and $\psi\left(x\right)$ is an eigenfunction of operator L^* corresponding to eigenvalues λ. More exactly $\overline{\psi\left(x\right)}$ is the solution of the equation

$$-z''+q\left(x\right)z=\lambda^2 z \tag{22}$$

belonging to $L_2\left(-\infty,+\infty\right)$. We obtained that $\psi\left(x\right)=0$, since the operator generated by expression standing at the left hand of (22), is an operator of type L. This contradiction shows that domain of value of the operator $\left(L-\lambda^2 I\right)$ everywhere dense in $L_2\left(-\infty,+\infty\right)$.
Lemma 3. The coefficient $C_{12}\left(\lambda\right)$ is an analytical function in the sector S_0 and there has finite number of zeros
Proof. For solutions $f_1^{\pm}\left(x,\lambda\right)$ and $f_2^{\pm}\left(x,\lambda\right)$ we can obtain the asymptotic equalities
$f_1^{\pm(j)}\left(0,\lambda\right)=\pm\left(i\lambda\right)^j+o(1)$ for $|\lambda|\to\infty,\ \ j=0,1,$
for $|\lambda|\to\infty,\ \ j=0,1.$
For simplicity we prove the first equality.
Since

$$f_1^{\pm}\left(0,\lambda\right)=1+\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{V_{n\alpha}}{n\pm 2\lambda}$$

that

$$\left|f_1^{\pm}\left(0,\lambda\right)\right|\leq 1+\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{|V_{n\alpha}|}{|n+2\lambda|}\leq 1+\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{|V_{n\alpha}|}{\sqrt{\left(n+2Re\lambda\right)^2+4Im^2\lambda}}\leq 1+\frac{1}{|Im\lambda|}\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{\alpha\,|V_{n\alpha}|}{n}.$$

Therefore, as $|\lambda|\to\infty$, we obtain $f_1^{\pm}\left(0,\lambda\right)=1+o\left(1\right)$.
Analogously we can prove the rest asymptotic equalities as $|\lambda|\to\infty$,
Then for the coefficients $C_{12}(\lambda),\ C_{11}(-\lambda),\ C_{12}(-\lambda),\ C_{11}(\lambda)$ we get the following asymptotic equalities

$$C_{12}(\lambda) = \frac{1}{2i\lambda}(\lambda\beta - i\lambda) + o(1) = -\frac{i\beta+1}{2} + o(1),$$

$$C_{12}(-\lambda) = -\frac{i\beta+1}{2} + o(1),$$

$$C_{11}(\lambda) = -\frac{1-i\beta}{2} + o(1),$$

$$C_{11}(-\lambda) = -\frac{1-i\beta}{2} + o(1).$$

These asymptotic equalities and analytical properties of the coefficients $C_{12}(\lambda),\ C_{11}(-\lambda),\ C_{12}(-\lambda),\ C_{11}(\lambda)$ make valid the following statement.

Lemma 4. The eigenvalues of operator $L$are finite and coincide with zeros of the functions $C_{12}(\lambda),\ C_{11}(-\lambda),\ C_{12}(-\lambda),\ C_{11}(\lambda)$ from sectors

$$S_k = \{{}^{k\pi}\!/_2 < \arg\lambda < {}^{(k+1)\pi}\!/_2\}, k = \overline{0,3}$$

respectively.

Definition 1. The data $\{\lambda_n, C_{11}(\lambda), C_{12}(\lambda)\}$are called the spectral data of$L$.

2.2. EIGENFUNCTION EXPANSIONS.

Definition 2. The points at which resolvent have poles are called the singular numbers of operator$L$.

Let $\lambda_1, \lambda_2, ...\lambda_l, \lambda_{l+1}.....\lambda_n$...be the singular numbers of operator L.At that

$$Re\lambda_j Im\lambda_j \neq 0, \quad j = 1, 2, ...l$$

$$Re\lambda_j Im\lambda_j = 0, \quad j = l+1, ...n, ...$$

Orderk_j of root$\lambda_j$is called the order of root of the singular numbers$\lambda_j,\ j = 1, ..l..$ It is clear that then the numbers $\lambda_{1,}\lambda_{2,}....\lambda_{l,}$ will be eigenvalues of operator$L$.The numbers $\lambda_j,\ j = l+1, ...n, ..$are called the spectral singularities of operator$L$. From the form of resolvent it is easy to see that it has singular numbers (i.e. eigenvalues) $\lambda_{1,}\lambda_{2,}....\lambda_{l,}$in zeros of the functions $C_{12}(\lambda),\ C_{11}(-\lambda),\ C_{12}(-\lambda),\ C_{11}(\lambda)$ in the sectors $S_k = \{{}^{k\pi}\!/_2 < \arg\lambda < {}^{(k+1)\pi}\!/_2\}, k = \overline{0,3}$ respectively. Their finiteness follows from Lemma 4. It directly follows from Lemma 2 and representation (17-20) that kernel of resolvent may have spectral singularities coinciding with the numbers$\frac{in}{2\beta},\ \frac{n}{2},\ n = \pm1, \pm2, \pm3, ...$ Consequently taking (15) into account and using

$$\lim_{\lambda \to {}^{n}\!/_2} (n - 2\lambda) W[f_2^+(0,\lambda), f_1^-(0,\lambda)] = V_{nn} W[f_2^+\left(0, \frac{n}{2}\right), f_1^+\left(0, \frac{n}{2}\right)]$$

we calculate

$$\lim_{\lambda\to n/2}(n-2\lambda)\,R_{11}(x,t,\lambda)=\lim_{\lambda\to n/2}(n-2\lambda)\,\tfrac{1}{2i\lambda}[f_1^+(x,\lambda)\,f_1^+(t,\lambda)\,\tfrac{W[f_2^+,f_1^-]}{W[f_1^+,f_2^+]}+$$
$$+f_1^+(x,\lambda)\,f_1^-(t,\lambda)]=\tfrac{1}{in}[V_{nn}f_1^+\left(x,\tfrac{n}{2}\right)f_1^+\left(t,\tfrac{n}{2}\right)+$$
$$+V_{nn}f_1^+\left(x,\tfrac{n}{2}\right)f_1^+\left(t,\tfrac{n}{2}\right)]=\tfrac{2}{in}V_{nn}f_1^+\left(x,\tfrac{n}{2}\right)f_1^+\left(t,\tfrac{n}{2}\right). \tag{23}$$

Analogously taking into account

$$\lim_{\lambda\to in/2\beta}(n+2i\lambda\beta)\,f_2^-(x,\lambda)=V_{nn}f_2^+\left(x,\frac{in}{2\beta}\right)$$

$$\lim_{\lambda\to in/2\beta}(n+2i\lambda\beta)\,W[f_2^-(0,\lambda),f_1^+(0,\lambda)]=V_{nn}W[f_2^+\left(0,\frac{in}{2\beta}\right),f_1^+\left(0,\frac{in}{2\beta}\right)]$$

we get

$$\lim_{\lambda\to in/2\beta}(n+2i\lambda\beta)\,R_{11}(x,t,\lambda)=\lim_{\lambda\to in/2\beta}(n+2i\lambda\beta)\,\tfrac{1}{2\lambda\beta}[f_2^+(x,\lambda)\,f_2^+(t,\lambda)\,\tfrac{W[f_2^-,f_1^+]}{W[f_2^+,f_1^+]}+$$
$$+f_2^+(x,\lambda)\,f_2^-(t,\lambda)]==\tfrac{1}{in}[V_{nn}f_2^+\left(x,\tfrac{in}{2\beta}\right)f_2^+\left(t,\tfrac{in}{2\beta}\right)+$$
$$+V_{nn}f_2^+\left(x,\tfrac{in}{2\beta}\right)f_2^+\left(t,\tfrac{in}{2\beta}\right)]=\tfrac{2}{in}V_{nn}f_2^+\left(x,\tfrac{in}{2\beta}\right)f_2^+\left(t,\tfrac{in}{2\beta}\right). \tag{24}$$

Calculate rest residues.

$$\lim_{\lambda\to -n/2}(n+2\lambda)\,R_{12}(x,t,\lambda)=\lim_{\lambda\to -n/2}(n+2\lambda)\,\tfrac{1}{2i\lambda}[f_1^+(x,\lambda)\,f_1^+(t,\lambda)\,\tfrac{W[f_1^-,f_2^-]}{W[f_2^-,f_1^+]}+$$
$$+f_1^+(x,\lambda)\,f_1^-(t,\lambda)]=\tfrac{1}{in}[V_{nn}f_1^-\left(x,-\tfrac{n}{2}\right)\ \tilde{f}_1^+\left(t,\tfrac{n}{2}\right)\tfrac{W[f_1^-\left(0,-\frac{n}{2}\right),f_2^-\left(0,-\frac{n}{2}\right)]}{W[f_2^-\left(0,-\frac{n}{2}\right),\tilde{f}_1^+\left(0,-\frac{n}{2}\right)]}+$$
$$+V_{nn}f_1^-\left(x,-\tfrac{n}{2}\right)f_1^-\left(t,-\tfrac{n}{2}\right)]=V_{nn}F_1\left(x,t,-\tfrac{n}{2}\right). \tag{25}$$

Here we use denotation $\tilde{f}_1^+(x,\lambda)=f_1^+(x,\lambda)\,(n+2\lambda)$, therewith, the function $\tilde{f}_1^+(x,\lambda)$ has no poles at the points $\lambda=-\frac{n}{2}$, $n\in N$, and $F_1\left(x,t,-\frac{n}{2}\right)=\frac{1}{in}[f_1^-\left(x,-\frac{n}{2}\right)\ \tilde{f}_1^+\left(t,\frac{n}{2}\right)\frac{W[f_1^-\left(0,-\frac{n}{2}\right),f_2^-\left(0,-\frac{n}{2}\right)]}{W[f_2^-\left(0,-\frac{n}{2}\right),\tilde{f}_1^+\left(0,-\frac{n}{2}\right)]}+f_1^-\left(x,-\frac{n}{2}\right)f_1^-\left(t,-\frac{n}{2}\right)]$.

Analogously we denote $\tilde{f}_2^+(x,\lambda)=f_2^+(x,\lambda)\,(n-2i\lambda\beta)$. Remark that the function $\tilde{f}_2^+(x,\lambda)$ also has no poles at the points $\lambda=-\frac{in}{2\beta}$, $n\in N$, and

$$F_2\left(x,t,-\tfrac{in}{2\beta}\right)=\tfrac{1}{in}[f_2^-\left(x,-\tfrac{in}{2\beta}\right)f_2^-\left(t,-\tfrac{in}{2\beta}\right)+$$
$$+\tfrac{W[f_2^-\left(0,-\frac{in}{2\beta}\right),f_1^-\left(0,-\frac{in}{2\beta}\right)]}{W[\tilde{f}_2^+\left(0,-\frac{in}{2\beta}\right),f_1^-\left(0,-\frac{in}{2\beta}\right)]}\tilde{f}_2^+\left(0,-\tfrac{in}{2\beta}\right)f_2^-\left(t,-\tfrac{in}{2\beta}\right)].$$

Then we get

$$\lim_{\lambda\to -in/2\beta}(n-2i\lambda\beta)\,R_{22}(x,t,\lambda)=-\lim_{\lambda\to -in/2\beta}(n-2i\lambda\beta)\,\frac{1}{2\lambda\beta}[f_2^-(x,\lambda)\,f_2^+(t,\lambda)\,\frac{W[f_2^-,f_1^-]}{W[f_2^+,f_1^-]}+$$
$$+f_2^+(x,\lambda)\,f_2^+(t,\lambda)]=\frac{1}{in}[V_{nn}f_2^-\left(x,-\frac{in}{2\beta}\right)f_2^-\left(t,-\frac{in}{2\beta}\right)+$$
$$+V_{nn}\frac{W[f_2^-\left(0,-\frac{in}{2\beta}\right),f_1^-\left(0,-\frac{in}{2\beta}\right)]}{W[\tilde{f}_2^+\left(0,-\frac{in}{2\beta}\right),f_1^-\left(0,-\frac{in}{2\beta}\right)]}\tilde{f}_2^+\left(0,-\frac{in}{2\beta}\right)f_2^-\left(t,-\frac{in}{2\beta}\right)]=V_{nn}F_2\left(x,t,-\frac{in}{2\beta}\right).$$

Lemma 6: Let $\psi(x)$ be an arbitrary twice continuously differentiable function belonging to$L_2(-\infty,+\infty,\rho(x))$. Then

$$\int_{-\infty}^{+\infty}R(x,t,\lambda)\,\rho(t)\,\psi(t)\,dt=-\frac{\psi(x)}{\lambda^2}+\frac{1}{\lambda^2}\int_{-\infty}^{+\infty}R(x,t,\lambda)\,g(t)\,dt,$$

where

$$g(t)=-\psi''(x)+q(x)\,\psi(x)\in L_2(-\infty,+\infty).$$

Integrating the both hand side along the circle $|\lambda|=R$ and passing to limit as $R\to\infty$ we get

$$\psi(x)=-\lim_{R\to\infty}\frac{1}{2\pi i}\oint_{|\lambda|=R}2\lambda d\lambda\int_{-\infty}^{+\infty}R(x,t,\lambda)\,\rho(t)\,\psi(t)\,dt$$

The function $\int_{-\infty}^{+\infty}R(x,t,\lambda)\,\rho(t)\,\psi(t)\,dt$ is analytical inside the contour, with respect to λ excepting the points$\lambda=\lambda_n,\ \lambda=\pm\frac{n}{2},\ \lambda=\pm\frac{in}{2\beta},\ n=1,2,....$ Denote by ${}^+_0\left({}^-_0\right)$ the contour formed by segments $[0,\frac{1}{2}+\delta],[\frac{n}{2}+\delta,\frac{(n+1)}{2}-\delta]$ and semi-circles of radius δ with the centers at points $\frac{n}{2},\ \ n=1,2,...$ located in upper (lower) half plane. Analogously we denote by the ${}^+_{0i}\left({}^-_{0i}\right)$ the contour formed by segments $[0,\frac{i}{2\beta}+\delta],[\frac{in}{2\beta}+\delta,\frac{i(n+1)}{2\beta}-\delta],\ \ n=1,2,3....$ and semi-circles of radius δ with the centers at points $\frac{in}{2\beta},\ \ n=1,2,...$ located in right (left) half plane.
Let the contours $\Gamma_1^+\left(\Gamma_1^-\right)$ and $\Gamma_{1i}^+\left(\Gamma_{1i}^-\right)$ be obtained from $\Gamma_0^+\left(\Gamma_0^-\right)$ and $\Gamma_{0i}^+\left(\Gamma_{0i}^-\right)$ by turning around the angleπ. Then

$$\psi(x)=-\frac{1}{2i\pi}\int_{-\infty}^{+\infty}2\lambda\rho(t)\,\psi(t)\,[\int_{\Gamma_{0i}^+}R_{11}(x,t,\lambda)\,d\lambda+\int_{\Gamma_0^+}R_{11}(x,t,\lambda)\,d\lambda-\int_{\Gamma_{0i}^-}R_{12}(x,t,\lambda)\,d\lambda+$$
$$+\int_{\Gamma_1^+}R_{12}(x,t,\lambda)\,d\lambda-\int_{\Gamma_1^-}R_{21}(x,t,\lambda)\,d\lambda-\int_{\Gamma_{1i}^-}R_{21}(x,t,\lambda)\,d\lambda+\int_{\Gamma_1^+}R_{22}(x,t,\lambda)\,d\lambda-$$
$$-\int_{\Gamma_0^-}R_{21}(x,t,\lambda)\,d\lambda]dt=-\frac{1}{2i\pi}\int_{-\infty}^{+\infty}2\lambda\rho(t)\,\psi(t)\,[\int_{\Gamma_{0i}^-}[R_{11}(x,t,\lambda)-R_{12}(x,t,\lambda)]d\lambda+$$
$$+\int_{\Gamma_0^-}[R_{11}(x,t,\lambda)-R_{22}(x,t,\lambda)]d\lambda+\int_{\Gamma_1^-}[R_{12}(x,t,\lambda)-R_{21}(x,t,\lambda)]d\lambda+$$
$$+\int_{\Gamma_{1i}^-}[R_{22}(x,t,\lambda)-R_{21}(x,t,\lambda)]d\lambda++\underset{\lambda=\lambda_n}{Res}[R_{11}(x,t,\lambda)+R_{12}(x,t,\lambda)+R_{21}(x,t,\lambda)+$$
$$+R_{22}(x,t,\lambda)]+\underset{\lambda=\frac{in}{2\beta}}{Res}R_{11}(x,t,\lambda)+\underset{\lambda=\frac{n}{2}}{Res}R_{11}(x,t,\lambda)+\underset{\lambda=-\frac{n}{2}}{Res}R_{12}(x,t,\lambda)++\underset{\lambda=-\frac{in}{2\beta}}{Res}R_{22}(x,t,\lambda)]dt$$

Separately calculate every item.

$$R_{11}(x,t,\lambda) - R_{12}(x,t,\lambda) = \frac{f_1^+(x,\lambda)\, f_1^+(t,\lambda)}{2i\lambda C_{12}(\lambda)\, C_{22}(\lambda)}$$

$$R_{11}(x,t,\lambda) - R_{22}(x,t,\lambda) = \frac{f_2^+(x,\lambda)\, f_2^+(t,\lambda)}{2i\lambda C_{12}(\lambda)\, C_{11}(\lambda)}$$

$$R_{12}(x,t,\lambda) - R_{21}(x,t,\lambda) = \frac{f_2^-(x,\lambda)\, f_2^-(t,\lambda)}{2i\lambda C_{11}(-\lambda)\, C_{12}(-\lambda)}$$

$$R_{22}(x,t,\lambda) - R_{21}(x,t,\lambda) = \frac{f_1^-(x,\lambda)\, f_1^-(t,\lambda)}{2i\lambda C_{22}(-\lambda)\, C_{21}(-\lambda)}$$

Residues of resolvent R_{pq} $p,q = 1,2$ in $\lambda_1,\lambda_2,....\lambda_l,$ denote by $G_{pq}(\lambda_n)$. Then $G_{pq}(\lambda_n)$ will be equal to

$$G_{pq}(\lambda_n) = \frac{1}{(k_n-1)!} \lim_{\lambda\to\lambda_n} \frac{d^{k_n-1}}{d\lambda^{k_{n-1}}} [(\lambda-\lambda_n)^{k_n} R_{pq}(x,t,\lambda)].$$

Then for every function $\psi(x)$ belonging to $L_2(-\infty,+\infty,\rho(x))$ we get following eigenfunction expansion in the form

$$\begin{aligned}
&\psi(x) = -\tfrac{1}{2i\pi} \int_{-\infty}^{+\infty} \rho(t)\,\psi(t)\,[\int_{\overline{0i}} [\tfrac{f_1^+(x,\lambda) f_1^+(t,\lambda)}{iC_{12}(\lambda) C_{22}(\lambda)}] d\lambda + \\
&+ \int_{\overline{0}} [\tfrac{f_2^+(x,\lambda) f_2^+(t,\lambda)}{iC_{12}(\lambda) C_{11}(\lambda)}] d\lambda + \int_{\overline{1}} [\tfrac{f_2^-(x,\lambda) f_2^-(t,\lambda)}{iC_{11}(-\lambda) C_{12}(-\lambda)}] d\lambda + \\
&+ \int_{\overline{1ii}} [\tfrac{f_1^-(x,\lambda) f_1^-(t,\lambda)}{iC_{22}(-\lambda) C_{21}(-\lambda)}] d\lambda + \sum_{\substack{p,q=1\\ \lambda=\lambda_n}}^{2} G_{pq} + V_{nn} F(x,t,n)] dt
\end{aligned}$$

where

$$F(x,t,n) = \frac{2}{in} f_1^+\left(x,\frac{n}{2}\right) f_1^+\left(t,\frac{n}{2}\right) + f_2^+\left(x,\frac{in}{2\beta}\right) f_2^+\left(t,\frac{in}{2\beta}\right) + F_1\left(x,t,\frac{n}{2}\right) + F_2\left(x,t,-\frac{in}{2\beta}\right)$$

SOLUTION OF THE INVERSE PROBLEM.

Let's study the inverse problem for the problem (1-3). The inverse problem is formulat as follows.
INVERSE PROBLEM. Given the spectral data $\{\lambda_n,\ C_{11}(\lambda),\ C_{12}(\lambda)\}$construct the$\beta$
and potential $q(x)$.
Using the results obtained above we arrive at the following procedure for solution of the inverse problem.
1. Taking into account (15) it is easy to check that

$$\lim_{\lambda\to{}^{n}\!/_{2}} (n-2\lambda) \frac{C_{11}(\lambda)}{C_{12}(\lambda)} = V_{nn},$$

consequently we find all numbers $V_{nn},\ n = 1,2,....$

2. Taking into account (6) we get

$$V_{n,\alpha+n} = V_{nn} \sum_{m=1}^{\alpha} \frac{V_{m\alpha}}{m+n},$$

from which all numbers $V_{n\alpha}, \;\; \alpha = 1, 2....., \;\; n = 1, 2, ...n < \alpha$ are defined.
3. Then from recurrent formula (6)-(8), find all numbersq_n.
4. The number β is defined from equality

$$\beta = iC_{11}(\lambda_n) C_{11}(-\lambda_n).$$

Really using Lemma 1 we derive for $\lambda_n \in S_0$ the true relation
$C_{11}(\lambda) = \frac{f_2^+(x,\lambda_n)}{f_1^+(x,\lambda_n)}$, $C_{22}(\lambda) = \frac{f_1^+(x,\lambda_n)}{f_2^+(x,\lambda_n)}$ i.e. $C_{11}(\lambda_n) C_{22}(\lambda_n) = 1$.
Then from (11) we get $\beta = iC_{11}(\lambda_n) C_{11}(-\lambda_n)$.

So inverse problem has a unique solution and the numbers β and q_n are defined constructively by the spectral data.

Theorem 2. The specification of the spectral data uniquely determines β and potential $q(x)$.

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